THE SYMMETRIC NATURE OF TRIANGLES

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In the process of some advanced geometrical investigations, we stumbled upon the following simple result about triangles. The result seems to be of independent interest, so rather than wait to include this as a nugget in a longer manuscript, we make it available here in a short note. As we found it a pleasant part of this investigation, we invite the reader to draw his or her own diagrams.

Theorem. All triangles are isosceles.

Proof. Let ABC be a triangle. Let D be the midpoint of segment BC. Let the perpendicular to BC at D meet the angle bisector of A at the point E.

Suppose first that E is inside the triangle.

Drop perpendiculars EF and EG from E to the sides of the triangle. Draw segments BE and CE. The triangles AEF and AEG have the side AE common and two angles congruent, so they are congruent by Euclid I.26 (AAS). Hence AF is congruent to AG and EF is congruent to EG. The triangles BDE and CDE have DE common, two other sides congruent, and the included right angles equal. Hence they are congruent by Euclid I.4 (SAS). In particular, BE is congruent to CE.

Now, the triangles BEF and CEG are right triangles with hypotenuses and a pair of legs congruent, so by Theorem on Hypotenuse-Leg for Right Triangles, they are congruent. Hence BF is congruent to CG. Adding equals to equals, we see that AB is congruent to AC, so that triangle ABC is isosceles.

There are other cases to consider. If the point E lies outside the triangle, one can use the same proof to conclude that AB and AC are the differences of equals, hence equal.

If E coincides with the point D, or if the angle bisector at A is parallel to the perpendicular to AB at D, the proof is even easier, and we leave it to the reader to complete these cases. □

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