Introduction

In this reading, we will see some new vocabulary to help us talk about graphs and their structure, and learn a theorem.

Goals

At the end of this assignment, a student should be able to:

- Discuss what makes a graph.
- Use the terms *isomorphism*, *planar*, *loop*, and *drawing* properly.
- State and use *Euler's Formula* appropriately.

A student might also be able to:

- Describe why there is no planar drawing of K_5, the complete graph on five vertices.
- Decide if the bipartite graph $K_{3,3}$ is planar or not.

Reading and Questions for Graph Theory Meeting Three

You may have noticed that mathematics uses a lot of specialized vocabulary. For someone new to a particular bit of math, this can be a challenge: there are so many new words it can be hard to keep them straight. So, this might seem like what makes mathematics hard. (It is sometimes.) But it is also what makes mathematics *possible*. Each vocabulary word is an *abstraction* — a whole bundle of ideas wound up into a single unit.

This reading has lots of vocabulary words in it. It is my belief that if you can see each word as connected to the ideas it represents, things will be easier. Right now, our most important vocabulary word is *graph*.

Some Technicalities About Graphs

We have decided that the word *graph* stands for a thing consisting of two related collections: first, we must have a collection of things, which we will call *vertices*; second, we must have a collection of relations between pairs of things, which we will call *edges*.

Here is one tricky bit. How can we use the idea of an edge? More specifically, can an edge be a relationship between a single vertex and itself, or must it connect two distinct vertices? Well this depends on the situation. Some people allow for edges which join a vertex to itself, and some do not. By the way, an edge which starts at a vertex and then comes back to that vertex is called a *loop*. (Try drawing a simple example. You will see that this is a good choice of term.) So, the question can be restated: Will the term *graph* allow loops or not?

There really is a choice to be made here. Sometimes allowing loops is helpful, sometimes it makes things more complicated. For our purposes, we just have to make a choice and live with it. So, I declare this: we will not allow loops. **A graph may not have an edge which starts and ends at the same vertex. That is, it may not contain loops.**

Exercise 1. Check that the complete graph on five vertices does not have any loops. (Hint: review the last reading if you do not recall what the complete graph on five vertices is.)

Connectedness and Components

Below is a drawing of a graph.

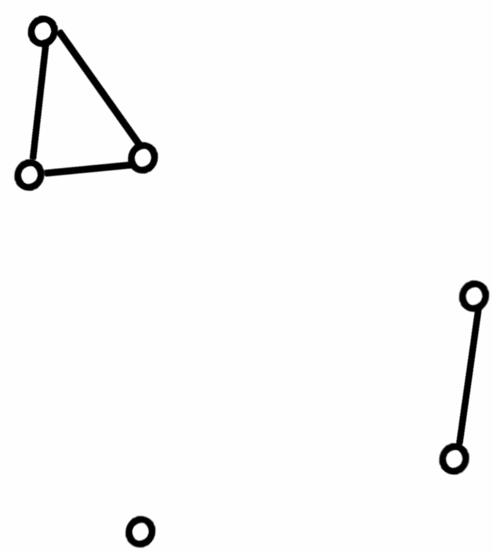


Figure 1: This is a single graph!

Do you find that odd in any way? Note that I said "a graph," not three graphs. But there are clearly three distinct pieces.

What is happening is that this graph fails to be *connected*. We say a graph is connected when for each pair of vertices, there is a chain of edges that you can use to go between that pair of vertices.

That fails in Figure 1 because we can choose some pairs of vertices with no such chain of edges. For example, the lone vertex near the bottom of the drawing has NO edges coming out of it at all, so it cannot be connected to any other vertex by a chain of edges.

This particular graph has three pieces, each of which is a sort of "maximally connected sub-graph." Each such piece is called a *connected component* of the graph (or just a *component* for short). Our graph has three components: one that looks like a triangle, one that looks like a line segment, and one that is a lonely vertex.

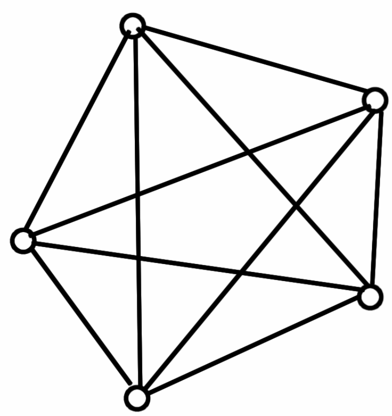
This is a spot which is easy to trip over. A graph can have lots of pieces (components), which makes it look like a collection of several graphs sitting next to each other. Be careful about this.

Exercise 2. Draw an example of a connected graph.

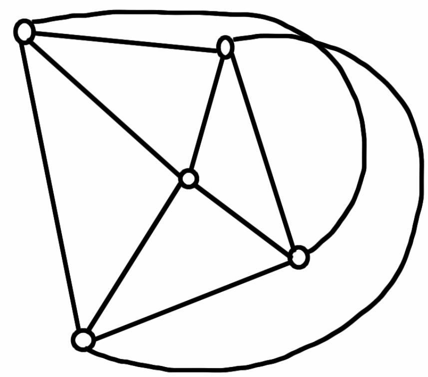
Exercise 3. Draw an example of a disconnected graph which is different from the one I drew in Figure 1.

Drawings and Isomorphisms

By now we have seen several examples of graphs. Here are two that you probably have seen before. (Careful, I am going to draw them next to each other to save space on the page, but this is really TWO graphs, not one graph with two components.)



(a) The standard drawing of K_5



(b) A non-standard drawing of K_5

Figure 2: Two drawings of K_5

Exercise 4. Convince yourself that these are really "the same graph."

What do we really mean by "the same graph?" Well, the idea is that the relationships represented by the graph Figure 2a are the same as the relationships represented by the graph in Figure 2b. In both cases, the graph tells us that there are five things, and each of the five things is related to each of the other four. In fact, each of these graphs is a different *drawing* of the complete graph on five vertices. This complete graph is important enough that mathematicians have given it a name. We call it K_5. (That is not quite as cool as BB-8.)

The kind of relationship where two graphs are really the same thing is called *isomorphism*, and we call the pair of graphs under consideration *isomorphic*. This funny word derives from the roots *iso*, meaning same, and *morph*, meaning shape or form.

To sum up, the graphs in Figure 2a and Figure 2b are isomorphic, and both are *drawings* of our special friend K_5.

Exercise 5. Pick a graph which is different from K_5. Make two different looking drawings of that graph.

Exercise 6. Draw two graphs which are **definitely not** isomorphic. Write down an explanation for why they are definitely different.

I hope it is clear to you that a single graph could have many different drawings, and those drawings will be isomorphic, but the way they are isomorphic might be really hard to see.

Planarity and Euler's Formula

We began class by looking for a way to draw K_5, the complete graph on five vertices, in a way that involved no crossing edges. This is pretty challenging. I hinted that not every graph has this property. This is important enough that it has a name.

A graph is called *planar* when it is possible to make a drawing of the graph which has no edge crossings.

If you recall, this is the same as saying that the graph has crossing number equal to zero.

Important Note to Dispel a Common Point of Confusion: There is a distinction to be made here between a graph, and the different drawings of that graph. A single graph has many different drawings, of course. The word planar gets applied to both, and we want to be careful. *A drawing is planar if it has no edge crossings.* You can see this just by looking. *A graph is planar if it has a drawing which is planar.* This is harder to check, because you have to either find a drawing which is planar, or somehow make an argument that there isn't any way to do that.

Exercise 7. Remember some examples of planar graphs that you have seen so far. If you do not remember any, go back through examples from the previous readings and homework, and note which ones are definitely planar.

Back at the dawn of graph theory, one of its first practitioners (Leonard Euler, pronounced "oil-er") found a really interesting fact about planar graphs. Nowadays, we write it down in a way to make it look official:

Theorem (Euler's Formula for Planar Drawings of Planar Graphs)**.** Suppose that we have a planar drawing of a planar graph. That is, a drawing of the graph where no edges cross. Let V be the total number of vertices in the graph, and let E be the total number of edges in the graph. The drawing of the graph divides the plane into a bunch of regions, including one region which lies completely outside the graph. Let R be the total number of regions. Then

$$V - E + R = 2$$

We're not going to prove this right now. Instead, let's look at an example.

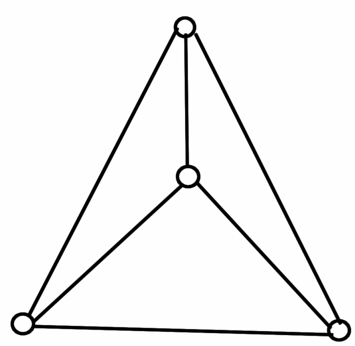


Figure 3: K_4, the complete graph on four vertices

For the graph in Figure 3, we have $V = 4$, $E = 6$ and $R = 4$ (don't forget the one on the outside!). So, we can check Euler's formula works here.

$$V - E + R = 4 - 6 + 4 = 2$$

Exercise 8. Make drawings of three different planar graphs, so that there are no crossing edges. Check that Euler's formula works for each.

Challenge. Can Euler's formula help you think about the Five Cities Puzzle? Or the Three Utilities Puzzle?

One Last New Name, and Some Hints

The graph in the Three Utilities Puzzle is called the *complete bipartite graph on three and three vertices*, and mathematicians call it by the name $K_{3,3}$. The idea here is that there are two groups of vertices, each with three vertices in it, and there are these edges: two vertices are connected by an edge exactly when they belong to different groups.

Here is a standard drawing of $K_{3,3}$.

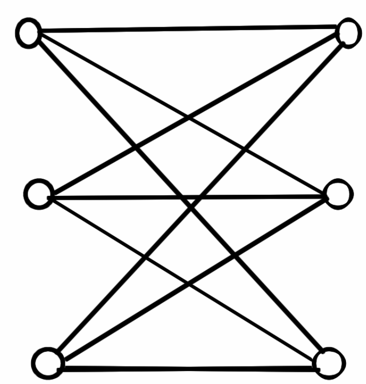


Figure 4: The complete bipartite graph graph $K_{3,3}$

The two puzzles we have considered so far can be restated in this way: Is K_5 planar? Is $K_{3,3}$ planar?

Challenge (Hint)**.** Suppose a graph has a triangle in it. Consider all of the possibilities for putting down two more vertices either inside the triangle or outside the triangle. Can this help you solve the Five Cities Puzzle?

Challenge (Hint)**.** The graph $K_{3,3}$ does not have any triangles in it. But you can find a square! What parts of what you did in the last challenge can be reused (adapted?) to the Three Utilities Puzzle?