Math in Decision Making
Unit on the Infinite: Matching, Counting, and Comparing Large Sets

Instructor: Theron J Hitchman

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Our main goal for the next work is to study some work of Georg Cantor on the nature of *infinity*. Have you thought about that before? It turns out that infinity is a rather slippery concept.

To make things clearer, we will have to practice speaking like mathematicians do. This will require *precision*. It will also require that we build up some new language–the language of *set theory*.

To keep things as "simple" as possible, we will mostly concern ourselves with things like counting, matching, and different types of numbers. Perhaps you have heard the terms *integer*, *rational number*, and *real number* before? Those await you in the following pages.

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Introduction

This assignment focuses on the basic language of set theory.

Goals

At the end of this assignment, a student should be able to:

- Demonstrate proper usage of the terms *set*, *element*, and *member*.
- Distinguish whether a particular object is, or is not, an element of a given set.
- Use set builder notation to describe a set.

A student might also be able to:

- Solve a challenging problem about some large sets.

Reading and Questions for Cantor's Paradise Meeting 2

In this unit, we shall study the concept of infinity. In particular, we shall explore different concepts of "number" and think about how big different collections of numbers are. Numbers began as a way to count things and keep track of how many things you have, so we shall encounter some challenges about counting things.

To talk about all of this clearly and coherently, we will have to be very precise about our use of language. The first task is to understand the concepts of *set*, and *element*.

Sets and Elements

The words *set* and *element* are undefined terms in mathematics. They are considered so foundational that we can't clearly say exactly what they are. Instead, we can just give some intuitive idea of them.

A *set* is a collection of some things. Those things in the collection are called the *elements* of that set. The only way to use these words is to say either

> The object x is an element of the set S.

or

> The object x is not an element of the set S.

Sometimes, the word *member* is used in place of *element*. In this context, *member* and *element* are synonyms. If we make this switch, our phrases read

The object x is a member of the set S.

or

The object x is not a member of the set S.

Membership is the only thing that matters with sets. Either your object is a member of the set, or it is not.

Example 1. We give two new examples of sets.

- The collection of all students enrolled at UNI is a set. Each enrolled student is a member of this set. Prof. Hitchman is not a member of this set. The chair on which you are sitting is also not a member of this set.
- The collection of all even integers is a set. We shall denote this set by the symbol $\mathcal{E}$. The numbers 4, 18, and 416 are all elements of $\mathcal{E}$, though not the only ones, by far. The number 227 is not an element of $\mathcal{E}$.

Exercise 1. Note that in the two examples above, one uses the terminology "member of" and the other uses the terminology "element of." Rewrite these sentences by hand, switching the terms around. (Yes. Seriously. Write it out. It will help.)

Notation for Sets

There are two very common ways to describe a set. The first way is just to use regular sentences. We did this in the two examples above. To make another example, let us first introduce the idea of an arrangement.

Definition. Suppose you have some things. An *arrangement* of those things is a particular way to make a list out of exactly those things in some order, without repeats and without skipping any.

For example, an arrangement of the athletic department mascots for UNI is

TC TK.

A different arrangement of those same mascots is

TK TC.

Example 2. Let T be the set of all arrangements of the letters A, B, and C.

Exercise 2. Write down three different elements of the set T.

Do not read any further until you have attempted that exercise.

Solution: What is an element of T? An arrangement of the letters of ABC is an element. Well, ABC is one arrangement. Then BAC is another one. And BCA yet another. Can you find a few more?

Exercise 3. Suppose that on a short trip you pack the following clothes in your bag:

- three shirts: red, blue, and green
- two pairs of pants: blue, and black-and-white checkerboard

Let J be the set of outfits you canmake by pairing a shirt with a pair of pants.

List all of the elements of J.

Exercise 4. Which is larger, T or J? Or are they the same size?

Another way to describe a set is by using "set builder notation." Here you give the name of the set, write an equality symbol, and then list the elements of the set inside a pair of curly braces.

Example 3. $S = \{3, 7, 12\}$ is a small set with three elements.

One thing that is a bit tricky is that your list really should not repeat. If it does, then the repeats do not count as new elements. Remember, all that matters is "Is x a member or is it not a member?" You cannot have an element that is a member more than once. Once is all you get.

Example 4. The sets

$$\{1, 1, 2, 3, 5, 8, 13, 21\}$$

and

$$\{1, 2, 3, 5, 8, 13, 21\}$$

are exactly the same thing. The number 1 is an element of both. The repeated 1 in the list for the first set is just a nuisance.

Some lists become too long to be convenient, so people often use an ellipsis $(\ldots)$ to imply that some pattern repeats to describe members of a set.

Example 5. The set of natural numbers is

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Example 6. The set $T = \{1, 4, 7, 10, 13, \ldots, 3001\}$ has implied members.

Exercise 5. Give examples of two elements of T that are not explicitly shown in the set builder notation.

Exercise 6. Give examples of two natural numbers which are **not** members of T.

Challenge. Let T be the set of all arrangments of the letters ABCDEFGHIJ. Let J be the set of all arrangments of the letters KLMNOPQRST. Write down some elements of T. Write down some elements of J.

Are these sets the same size? Or is it the case that one is larger than the other? If one is larger? Which one?

Introduction

In this assignment, you will learn about *subsets*. This will give us flexibility to build many new examples and describe them carefully. Also, you will learn about a special construction, the *power set* of a set.

Goals

At the end of this assignment, a student should be able to:

- Describe clearly the meaning of the word subset.
- Decide if one set is, or is not, a subset of another.
- Model some counting problems using the language of subsets.

A student might also be able to:

- Show that some counting problems involving subsets are very similar.

Reading and Questions for Cantor's Paradise Meeting 3

The Notion of a Subset

Suppose you have a crowd of 50 people, and from that crowd you must choose two people to be representatives for a mission to Mars. How many ways can you make such a choice? The number is rather large, but for now we do not care what it is. More important to us is the structure of what is going on.

Initially, you have a set of 50 people like this:

$$C = \{\text{Jane}, \text{Joe}, \text{Jessica}, \text{Jeremey}, \dots, \text{Jennifer}\}.$$

(Recall that the ellipsis, that is the three periods in a row, indicates that some elements are missing from the description.) A choice of a pair of people to go to Mars is the a set like $\{\text{Jane}, \text{Joe}\}$ or $\{\text{Jeremey}, \text{Jennifer}\}$. Each of those things is a *subset* of the original set C. What does this mean?

Definition. Let S be a set. Another set R is called a *subset* of S when for each element of R, that thing is also an element of S.

Why is $\{\text{Jane}, \text{Joe}\}$ a subset of C? Just check! The elements of $\{\text{Jane}, \text{Joe}\}$ are Jane and Joe. Each of these is an element of C, also. Therefore, by the definition above, $\{\text{Jane}, \text{Joe}\}$ is a subset of C.

Exercise 1. Give two other examples of subsets of C and describe how you know they are subsets.

Exercise 2. Find two more subsets of C, this time one of your subsets should have less than two elements, and one should have more than two elements.

Some Exercises

Exercise 3. Consider the set $X = \{A, 3, \{1\}\}$. List as many subsets of X as you can. (There are eight of these. How many can you find?)

That last exercise is tricky. We will come back to count all of the subsets in just a bit.

Exercise 4. Let $T = \{1, 2\}$ and $J = \{1, 3\}$ be two sets. Use the definition to write some sentences that explain why T is not a subset of J and J is not a subset of T.

Constructions of subsets allows us great flexibility but it is important to be careful. For the next few exercises, we will work with the set G,

$$G = \{\{1, 2\}, \{3, 7\}, \{1, 3\}\}.$$

Exercise 5. Use the definition of subset to explain why $H = \{\{1, 2\}, \{1, 3\}\}$ is a subset of G.

Exercise 6. Use the definition of subset to explain why $K = \{1, 2, 3\}$ and $B = \{7\}$ are not subsets of G.

Oddities of Subsets

Mathematicians have found it useful to make some conventions that seem odd to the newcomer.

The Empty Set

First, there is a special set called the *empty set*. This is the set that *has no elements*! The standard way to write this is with this symbol $\emptyset$, but really that is just shorthand for $\{\quad\}$.

The goofiest thing about the empty set is that it is a subset of every other set. Why? Think about the definition: for each element of $\emptyset$ we need something to be true. But there are no elements of $\emptyset$. So we get that whatever we need is true because there is nothing to check.

The Whole Subset

Suppose you have some set S. If you work through the definition of the word subset, you can see that S is a subset of itself. This is a bit weird because our intuition usually says that a subset should be something smaller than the original. But this is the way the definition works. Any set is a subset of itself.

Exercise 7. Go back to the exercise about finding all of the subsets of $X = \{A, 3, \{1\}\}$. Can you find all eight of them now?

The Power Set of a Set

Collecting subsets together allows us to make an important construction.

Definition. Let S be a set. The *power set of* S is the set whose elements are all of the subsets of S.

The power set is usually denoted by $\mathcal{P}$. So, in set builder notation, this definition looks like this:

$$\mathcal{P}(S) = \{X \mid X \text{ is a subset of } S\}$$

Here we have extended the set-builder notation scheme. Inside the curly braces, you find a description with some name in it (here it is X), then a vertical bar, and then a description of the rule one uses to decide if the name belongs to the set or not.

Exercise 8. Let $F = \{1, 2, 3, 4\}$ be a set. Describe the power set $\mathcal{P}(F)$ of F by writing down all of its elements. Note that there are sixteen of them.

Challenges

Here are some tasks that will help you practice common things we will want to do with subsets.

Challenge. In a city of 4000 people, we are to choose some to get shiny gold medals with pictures of Prof. Hitchman's distinguished face on one side, and the number 403 on the other. We want to pick three special people to get medals.

(a) Make a definition of a set T so that its elements represent possible choices of three people to *get* medals. You should use the word "subset" somehow.

(b) Make a definition of a set J so that its elements represent possible choices of people to *not* get medals. You should use the word "subset" somehow.

(c) [Extra challenge] How can we see that T and J have the same number of elements without counting?

Challenge. Let $\mathbb{N}$ be the set of natural numbers, and let $\mathcal{E}$ be the set of all even natural numbers.

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}, \qquad \mathcal{E} = \{2, 4, 6, 8, \ldots\}$$

Use the definition of subset to check that $\mathcal{E}$ is a subset of $\mathbb{N}$, but $\mathbb{N}$ is not a subset of $\mathcal{E}$.

Which of these sets is bigger?

Introduction

In this reading, we learn about the idea of a matching. This is a way to compare the sizes of two sets without actually counting them.

Goals

At the end of this assignment, a student should be able to:

- Describe the idea of a matching between two sets.
- Use the idea of a matching between two finite sets to show that those sets have the same number of elements.
- recognize a common pitfall in constructing matchings.

A student might also be able to:

- Solve a challenging problem using a matching between sets.

Reading and Questions for Cantor's Paradise Meeting 4

Mathematicians want to understand the relative sizes of very large sets. But it can be difficult to count the number of elements of a large set. Worse, it could be inconvenient. But there is a way to keep track of the idea of two sets "having the same size" without actually knowing what the size is! This is our goal for today.

The Idea of a Matching: A Parable?

Have you ever tried counting to a big number really fast? It gets challenging as the words pile up. *One hundred seventy-seven* is a lot of syllables.

Imagine a shepherd with a large flock. The sheep move in and out of the pen a bit too fast for him to count, as the words describing the numbers get to be a mouthful. Instead, he gets a bag of pebbles and makes sure he has one pebble for each sheep. How many pebbles is it? It doesn't matter. What matters is that he can match them up with the sheep. Each morning, he moves one pebble for each sheep that passes out of the gate into a second bag. If the last sheep goes out as the last pebble switches bags, he has seen them all. This can be repeated in the evening. If the last sheep comes in as the last pebble changes from one bag to the other, then all of the sheep have come home. If the sheep and the pebbles do not match, he knows if any sheep are missing, or if there is somehow a new addition to the flock.

This is the idea of a matching. In this case, the flock of sheep is one set (whose elements are the individual sheep), and the bag of pebbles is the other set (whose elements are the individual pebbles). Each morning and each evening, the farmer is observing a matching between the elements of these two sets.

Definition. Let A and B be two sets. A *matching between the elements of A and the elements of B* is a way to associate to each element of A exactly one element of B, so that no elements of either set are left unpaired.

Example 1. Let A be the set of fingers on your right hand. Let B be the set of fingers on your left hand. It is usually the case that these two sets have the same size. The most common matching between the elements of A and the elements of B is made by pressing your fingertips together like Mr. Burns from *The Simpsons*. **Excellent!**

Exercise 1. Let T be the set of all arrangements of the letters $ABCDEFGHIJ$. Let J be the set of all arrangements of the letters $KLMNOPQRST$. First, be sure to figure out what counts as an element of T and what counts as an element of J. (Sanity Check: There are lots of elements of both sets. Way more than 100.)

Find a matching between the elements of T and the elements of J. Use this to describe how you know that these two sets have the same size, even though you don't know what that size is.

Exercise 2. Let $T = \{1, 4, 7, 10, 13, \ldots, 3001\}$. Let $J = \{2, 5, 8, 11, 14, \ldots, 3002\}$. Describe a matching between the elements of T and the elements of J.

As a check: can you see why these two sets both have 1001 elements?

One Common Pitfall

Even after you get the basic idea of a matching, it pays to be careful. There are a few common mistakes one can make. One of them is encapsulated in the next exercise.

Exercise 3. My friend Penny is very young, and is still learning to count. When she first tried it, she often counted things like this:

> One, Three, Four, Five, Eight, ...

What is the nature of Penny's mistake?

Can you use the language of matchings to say exactly what Penny's mistake is?

A Challenge

Challenge. Horror of horrors, you are in charge of every third grader in Cedar Falls for an afternoon. Fortunately, you have them all trapped in the UNI-Dome. They are all running around on the field shouting, "Look at me! I'm a vampire unicorn!"

Quick, without counting, come up with a way to decide if there are more boys, more girls, or the same number of each.

Introduction

This assignment is a further discussion of the idea of a matching. We examine other common pitfalls encountered when making matchings, and how matchings can be used to make comparisons.

Goals

At the end of this assignment, a student should be able to:

- Describe the idea of a matching clearly in plain language.
- Describe two common mistakes made when making matchings.
- Use the idea of a matching to make comparisons between the sizes of sets.
- Construct matchings between sets.

Reading and Questions for Cantor's Paradise Meeting 5

Let us continue our examination of the idea of a matching between the elements of two sets.

The Definition of a Matching.

In the last reading, we learned an official definition of a matching between the elements of two sets. It looks like this:

Definition. Let A and B be two sets. A *matching between the elements of* A *and the elements of* B is a way to associate to each element of A exactly one element of B, so that no elements of either set are left unpaired.

A tricky thing here is that the matching is *the association.* Another way to describe this is that a matching is a *rule for assignment*: for each element a of the set A, the rule tells us how to find an particular element b of the set B. These two elements a and b are then paired together. The part of the definition which says that for each member a of the set A there must be exactly one associated member b of B, and vice versa, expresses the idea of a *one to one correspondence.*

Example 1. Let $T = \{1, 4, 7, 10, 13, \ldots, 3001\}$. Let $J = \{2, 5, 8, 11, 14, \ldots, 3002\}$. We now describe a matching between the elements of T and the elements of J.

Given an element of T, we can associate to it the element of J which is one greater. If you want, you can describe this with algebraic symbols like this:

> To each element x of T we associate the element $x + 1$ of J.

But, really, all we are doing is specifying the following rule for assignment. If you give me an element of T, I find the associated element of J by **adding one**.

Note that this rule is completely reversible. If we start with an element of J, we can find the associated element of T by **subtracting one**. This is the way that a matching between elements is just a rule for assignment. We give the instructions for how to find each elment's pair-partner in the other set.

Exercise 1. Explain in sentences how you can be sure that this matching between the elements of T and the elements of J are in a one to one correspondence.

Common Mistakes

Think about counting. Counting is a form of making a matching between those things you wish to count and some subset of the natural numbers. In particular, to count a collection of things is the same as making a matching of those things with some "initial subset" of $\mathbb{N}$. When counting a baby's toes (certainly one of the most fun things you can do), you are really making a matching between the set {toes on this baby} and the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

In the last reading, we described a common error in the business of making matchings.

Example 2. My friend Penny is very young, and is still learning to count. When she first tried it, she often counted things like this:

> One, Three, Four, Five, Eight, ...

Penny's mistake is that she is skipping over some numbers. Thus the construction of a matching between what ever she is counting and the set of natural numbers is faulty. There will be relevant natural numbers which don't have pair-partners! Essentially, nothing gets labeled "two," so the matching is doomed.

Exercise 2. My little friend Penny no longer makes that mistake of skipping numbers. Instead, she makes a different mistake. Now she counts like this:

> One, two, three, four, five, six, seven, eight, nine, ten, eleven, eight, nine, ten, eleven, eight, nine, ten, eleven, ...

(It can go on like this for a long time. You get the idea.) What is the nature of Penny's second mistake?

Use the language of matchings between elements to describe what is going wrong.

Making Comparisons

An important feature of having a matching between the elements of one set and the elements of another set is that it means that in a strict sense those two sets have the same size. This is nice, because it gives us a way to say things are "the same" directly, without having to go through the intermediate step of counting those sets. (Counting really large sets is often inconvenient.)

Often, the ways in which we fail when attempting to make matchings can give us clues to the relative sizes of two sets. The precise way in which we cannot produce a one to one correspondence might tell us which of the two sets is bigger!

For example, In class we saw that the set C of chairs in the classroom had more elements than the set S of students. We observed this because having everyone sit down was a form of attempting to make a matching between the elements of S and the elements of C. Each student (an element of S) associated himself or herself with a chair (an element of C) by sitting in it. That is a rule for the assignment!

Exercise 3. Which of two of Penny's mistakes are made in trying to form this matching?

Exercise 4. Suppose that somehow our assignment of students to chairs failed to be a matching because it made the other mistake Penny taught us about. What would be happening, physically, with students and chairs?

Exercise 5. Suppose that we have two sets where A is definitely larger than B. Why is it impossible to make a matching by coming up with a rule of association that starts with elements of A and produces elements of B?

Exercise 6. Suppose that we have two sets where A is definitely larger than B. Why is it impossible to make a matching by coming up with a rule of association that starts with elements of B and produces elements of A?

Introduction

In this reading we reassess the idea of counting and compare it to our idea of matching.

Goals

At the end of this assignment, a student should be able to:

- Explain how counting is making a matching with some initial segment of the natural numbers.
- Explain how counting compares with making a list.
- Explain the importance of the order given to the elements of the list.

A student might also be able to:

- Solve some challenging problems about counting by using lists appropriately.

Reading and Questions for Cantor's Paradise Meeting 6

So far, we have tried to learn to compare the sizes of sets *without* actually counting them. The idea of comparing sets by attempting to match them up is an old and powerful one, but of course it is not the only one. Let us revisit the idea of counting with a fresh perspective.

A New Look at Counting

Begin with the following simple exercise. Pay attention to *how* you solve it.

Exercise 1. Consider the set $C = \{\pi, e, 7, 22, 948, 51, 403\}$. How many elements does C have?

What did you do when you completed that exercise? I am sure you counted the elements. Did you touch each element of C with a pencil while reciting the names of numbers in the proper order? Maybe you did not physically touch the elements, but instead just let your eye linger over each as you said the names of numbers in your head?

> *One, two, three, four, five, six, seven.*

What you just did was to make a matching between the elements of C and the set $\{1, 2, 3, 4, 5, 6, 7\}$. You are not accustomed to thinking of it that way, but that is what happened.

This is how counting works! When we teach children to count, there are really two things happening.

(Obvious) We help children commit to memory our standard scheme for naming numbers in a particular order.

(Not obvious) We teach children the implicit process of making a matching between the collection of things to be counted and some properly chosen subset of the natural numbers.

The second part lies deeper in the process. Even though it is usually done implicitly, children pick it up at some point. I know a four year old who has the process picked up, she always makes matchings, now. But she has not, yet, mastered the scheme for naming numbers. In a way this is reasonable, the naming scheme has some inherent order to it, but it really is arbitrary. (For elementary school children a lot of instruction goes on about the place-value system which underlies the Hindu-Arabic numeral system we all use.)

Example 1. Have you ever played with a toddler? One fun and educational activity is to put the child on your lap and count the child's toes. This helps the child learn about the numbers *one* through *ten*, but also about the matching process. As you count, you touch the child's toes individually, and thus embody the matching.

From our more advanced perspective, what is happening? Recall that $\mathbb{N}$ denotes the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Definition. Fix a natural number n. The *initial segement* of $\mathbb{N}$ of length n is the set of all natural numbers k which are no greater than n. We shall denote the initial segment of length n by the symbol $\lfloor n \rfloor$.

For example, the initial segment $\lfloor 10 \rfloor$ is the set

$$\lfloor 10 \rfloor = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Observation 1. Counting the elements of a set X is the same thing as making a matching between the elements of X and the elements of some initial segment $\lfloor n \rfloor$ of the natural numbers. The number n is what we usually call the number of elements of X.

The Importance of Ordering

So, how can we use this to further our goals? The idea lies in this exercise:

Example 2. How many one letter mathematical words are there?

More carefully, let W_1 be the set of all mathematical words on the usual alphabet of uppercase letters. How many elements does W_1 have?

You probably know that there are 26 letters. But if you didn't know, you would make a list of all of the uppercase letters. Out of habit, I bet you would put them in the usual order, too.

$$W_1 = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\}$$

This particular list of elements, with its ordering, makes a matching of the elements of W_1 with the elements of $\lfloor 26 \rfloor$.

The ordering is critical here! For example, what is the 16th element? It is P. This means that P in W_1 is paired up with 16 in $\lfloor 26 \rfloor$.

Observation 2. Making a list of the elements of some set is exactly the same thing as constructing a matching between the elements of that set and some initial subset of the natural numbers.

The matching rule is implicit in the list in the following way: *The ordering of the elements of the list tells us how to match them with numbers.* The first element in the list gets matched with 1, the second element in the list gets matched with 2, and so on.

Now, you try using this idea.

Exercise 2. Let W_2 be the set of all mathematical words with two letters chosen from the alphabet of uppercase letters. Find a way to make a list of all the elements of W_2. Use your list to explain why there is a matching between the elements of W_2 and the elements of $\lfloor 676 \rfloor$. (Note that $676 = 26 \times 26$.)

To recap, the key ideas are these:

- Counting is really an example of a matching process. You match the elements of the set you wish to count with some special subset of $\mathbb{N}$ called an initial segment.
- A matching with an initial segment of the natural numbers is the same thing as making a list of the elements, because the ordering of the list holds the matching rule.

Challenges

To keep sharp, think about how the ideas we have learned can help you solve these problems.

Exercise 3. Make a list that shows how the set of all arrangements of the digits 1, 2, 3 and 4 has a matching with the set $\lfloor 24 \rfloor$.

Exercise 4. Let $Three = \{3, 6, 9, 12, 15, 18, \ldots\}$ be the set of all natural numbers which are multiples of 3. Describe how there is a matching between the elements of $Three$ and the elements of $\mathbb{N}$ hiding in this description of $Three$.

Introduction

In this reading, we make precise the difference between finite and infinite sets.

Goals

At the end of this assignment, a student should be able to:

- State clearly what it means for a set to be finite.
- State clearly what it means for a set to be infinite.
- State clearly what it means for a set to countably infinite.

Reading and Questions for Cantor's Paradise Meeting 7

Now that we have some comfort with comparing sets via matching elements, and with counting, it is time to make distinctions between sets which are merely big and those which are *big*.

Finite Sets

What does it mean for a set to be finite? At some intuitive level, it means that we can start listing all of them, and at some point that list will stop as we have considered everything in the set. This can be made precise with the following definition.

Definition. Let S be a set. We say that S is a *finite* set when there is some natural number n, and a matching between the elements of S and the elements of the initial segment $\lfloor n \rfloor$ of the natural numbers.

Example 1. Let $X = \{0, 1\}$. The power set of X, $\mathcal{P}(X)$ is finite. To see this, we make a list of the elements of $\mathcal{P}(X)$ as follows:

$$\mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, X\}.$$

The ordering of this list makes an implicit matching with the initial segment $\lfloor 4 \rfloor = \{1, 2, 3, 4\}$ of the natural numbers.

Exercise 1. Let $Y = \{0, 1, 2\}$. Show that the power set $\mathcal{P}(Y)$ of Y is a finite set.

Most sets that people encounter in everyday life are finite. Even if they are really big, they are still somehow understandable as just a list of things. Of course humans are not generally good at understanding the true nature of large numbers. It is very difficult to get an accurate impression of how much bigger the number 1000000000000 is than the number 1000.

Infinite Sets

The word *infinite* is the opposite of *finite*. It literally means "not finite." So if we invert the definition above we get this:

Definition. Let S be a set. Assume that S is not the empty set. We say that S is an *infinite* set when it is impossible to choose a natural number n and a matching between the elements of S and the elements of the initial segment $\lfloor n \rfloor$.

That definition is a bit harder to use. First, it is harder to read. But read it a couple of times and it will sink in. Second, it states an *impossiblity* rather than a *possibility* as the condition to check. That is much harder to deal with.

Fortunately, we have this theorem which makes our life easier.

Theorem. Let S be a set. Then S is infinite if, and only if, there is a subset T of S such that one can make a matching between the elements of T and the elements of N.

How does this help? Well, it gives us something positive to check. Instead of having to check a statement like "there is no such thing as a...", we can check a statement like "it is possible to make..."

Note that the theorem has the special words "if, and only if" in it. This is mathematician's code for *these two statements are equivalent.* So if we have one, then we have the other, and vice versa.

I won't give a formal proof of the theorem, because it is a bit ugly. The basic idea is to consider lists of elements of S. Just start making such a list and see what happens. If your set is not finite, then any list *has to keep going on forever*, because a list which stops is exactly our definition of the word finite.

And in the other direction, if you have a list of the elements of T which goes on forever, then clearly no list which stops will be good enough to match all of those elements. When you try to make the matching between elements of some initial subset $\lfloor K \rfloor$ of the natural numbers and the elements of T, you will necessarily miss a lot of them.

Example 2. The set $\mathcal{W}$ of all mathematical words of any length is an infinite set.

To see this, consider the special subset $\mathcal{W}_A$ consisting of all of those words whose only letter is A. We may list the elements of $\mathcal{W}_A$ as follows

$$\mathcal{W}_A = \{A, AA, AAA, AAAA, AAAAA, AAAAAA, \ldots\}$$

This list makes a matching with the set $\mathbb{N}$ of natural numbers. We conclude that $\mathcal{W}_A$ is infinite.

Exercise 2. The set of *integers* is the set consisting of all of the natural numbers, their negatives and zero. A common notation for this set is the letter $\mathbb{Z}$.

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Give an argument for why this set is infinite by using the theorem above.

Countably Infinite Sets

There is a special collection of infinite sets that have the property of being exactly the same size as the natural numbers. Such sets are called countably infinite sets.

Definition. A set X is called *countably infinite* when there exists a matching between the elements of X and the elements of $\mathbb{N}$.

We have previously encountered the set $\mathcal{E}$ of even natural numbers, and the set $\mathcal{O}$ of odd naturnal numbers. Each of those sets is countably infinite.

Exercise 3. Show that the set of all natural numbers which are evenly divisible by four is countably infinite.

Introduction

Here, you pause to take stock and think the context and structure of what you have learned.

Goals

At the end of this assignment, a student should be able to:

- Make an outline of the topics we have studied so far.
- Describe how our study of sets, matching, and counting adds up.

Reading and Questions for Cantor Meeting 8

It is time to reflect a little bit. After three weeks, you have learned a lot.

Exercise 1. Reread the previous READING AND GUIDED PRACTICE assignments.

Exercise 2. Make an outline of what you have learned so far. How are the topics related? What are the major points of which to keep track? Have you discovered any new ways of thinking you should note for later?

Introduction

We introduce the idea of a proof by contradiction. This is used to prove that the set of prime numbers is an infinite set.

Goals

At the end of this assignment, a student should be able to:

- State clearly what what a "proof by contradiction" is.
- Describe the logic behind a proof by contradiction.
- Explain why the set of all prime numbers is an infinite set.

Reading and Guided Practice for Cantor's Paradise Meeting 9

Mathematicians have known for a long time about *prime numbers*.

Definition. A natural number p which is greater than 1 is called a *prime number* when its only divisors are 1 and p.

Example 1. The first few prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \dots$$

What makes prime numbers so important? They are like little atoms for multiplication. They are the smallest indivisible pieces of natural numbers.

Our goal today is to understand the argument for the following important fact, which has been known since at least the ancient Greek culture mathematicians. (This work appears in Euclid's *Elements*.)

Theorem (The Infinitude of Primes)**.** The set $\mathbb{P} = \{p \mid p \text{ is a natural number and } p \text{ is prime}\}$ of all prime numbers is an infinite set.

Exercise 1. What is the next largest prime number after 31?

A Preface on the Argument

We are going to look at the proof of that theorem. It is not necessary to be afraid of the word *proof* here. All that we mean by the word *proof* is that we are going to give a convincing argument about why the theorem is true.

But in this case, the convincing argument has a clever technique in it. The argument is an example of a *proof by contradiction*. What happens is this:

Step 1: We begin by assuming the theorem is **false**.

Step 2: We use that assumption to argue that some other thing happens.
Step 3: The trick is that the "other thing" has to be obviously wrong.
Step 4: Since our assumption led us to something stupid, we conclude that our assumption must be incorrect.
Step 5: Therefore our theorem must be true.

This is the logic behind a proof by contradiction. It can be easy to get things turned around a bit when you are new to this type of argument, because you have to assume negative things a lot. People who are used to this kind of argument spend all their time on Step 2, and they do not usually explain that the other steps are happening, too. But after a bit of practice, you will find this natural and all will be well.

Exercise 2. Have you ever seen a proof by contradiction before? Try to think of one or two situations where it has come up.

Exercise 3. How is a proof by contradiction like a jury trial? (This is a weak analogy, but there is something to be learned here.)

A Fact we need for the proof

To make the proof run, we need a fact which you likely already know. So that things are clearest, we state it right now up front as some thing you can and should believe. But so we do not get distracted, we will not give a proof of this statements.

The Fundamental Theorem of Arithmetic

Theorem. If n is a natural number, then there is a unique way to write n as a product of prime number factors.

There are actually two things happening here. First, every natural number can be written as a product of prime numbers. For example, $6 = 2 \cdot 3$, and $100 = 2 \cdot 2 \cdot 5 \cdot 5$.

Exercise 4. Write the number 204 as a product of prime number factors.

Second, there is only one way to write a given number as a product of prime numbers. The only thing you can do is rearrange the list of primes so it reads in a different order.

For example, the only interesting thing one can do with the expression for 6 is reorder it like this: $6 = 3 \cdot 2$.

A Proof of The Infinitude of Primes

We shall make an argument by contradiction. Suppose that the set $\mathbb{P}$ is not infinite. That is, assume that $\mathbb{P}$ is a finite set.

Since $\mathbb{P}$ is finite, there must be some list of primes which stops at some point. We write this list as a description of $\mathbb{P}$ as follows:

$$\mathbb{P} = \{p_1, p_2, p_3, p_4, \ldots, p_{n-1}, p_n\}.$$

We do not know what the length n is, or what each individual prime p_i in our list is. That will not matter! What matters is that the list stops.

Here is a big leap. Consider the new number

$$X = p_1 \cdot p_2 \cdot \dots \cdot p_{n-1} \cdot p_n + 1,$$

which is made by multiplying together all of the prime numbers on our list, and then adding 1. Note that X has to be bigger than all of the numbers in our list of prime numbers.

By the Fundamental Theorem of Arithmetic, X can be written as a product of prime numbers. Which prime numbers are factors of X? We shall check each of the prime numbers in our list.

Does p_1 divide X? No. When you try to divide X by p_1, you will get a remainder of 1.

Does p_2 divide X? No. When you try to divide X by p_2, you will get a remainder of 1.

This little argument works *for all of the prime numbers on our list*!

Since X has to have at least one prime factor, we see that this prime factor is some new, unknown number which was not on our list! This is a problem. All of the prime numbers are in our list.

So, something about the above is wrong. What is wrong? Our assumption at the beginning.

We conclude that the set $\mathbb{P}$ of all prime numbers is an infinite set.

1 Some Exercises

When confronted with a new argument, it can be helpful to work out several special cases to see what happens. This particular argument can almost be used as a recipe for trying to find new prime numbers. When you do so, you quickly find this sequence of tasks.

Exercise 5. Run through the argument in the specific case where $\mathbb{P} = \{2, 3\}$. See that the number X in this case is a prime number.

Exercise 6. Run through the argument in the specific case where $\mathbb{P} = \{2, 3, 7\}$. See that the number X in this case is a prime number.

Exercise 7. Run through the argument in the specific case where $\mathbb{P} = \{2, 3, 7, 43\}$. See that the number X in this case is not prime. What are the prime number factors of this X? Are they on the list $\mathbb{P}$?

Introduction

We reacquaint ourselves with old friends, the *rational numbers*.

Goals

At the end of this assignment, a student should be able to:

- Describe clearly what a rational number is.
- Recognize when two rational numbers are equal.
- Recognize when one rational number is greater than another.
- Discuss the ordering of rational numbers, and how it differs from the ordering of the natural numbers.

A student might also be able to:

- Show that the set of rational numbers is an infinite set.

Reading and Questions for Cantor's Paradise Meeting 10

Recall that we have encountered the set of *natural numbers*

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

and the set of *integers*

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

Now we will introduce a new kind of number and consider the set of all of them.

A *rational number* is one of the form a/b where a is any integer and b is any natural number. For example, $-4/7$ comes from choosing $a = -7$ from $\mathbb{Z}$ and $b = 7$ from $\mathbb{N}$.

Exercise 1. Make three more examples of rational numbers and say what the choices of a and b are.

The collection of all rational numbers is denoted $\mathbb{Q}$. (Here the funny Q is for "quotient," since we make rational numbers by making a quotient of a by b.)

The first interesting thing to work out about rational numbers is that some of them are equal without it being immediately obvious! For example, $1/2$ is really the same thing as $2/4$. Typically, these are called *equivalent* rational numbers.

Definition. Two rational numbers a/b and a'/b' are *equivalent* exactly when $ab' = a'b$.

Note the multiplication is just ordinary multiplication of integers like in grade school.

Exercise 2. List three pairs of equivalent rational numbers.

Exercise 3. List three pairs of inequivalent numbers.

Comparing Rational Numbers

One nice thing about the natural numbers is that we can arrange them in a line. Like an infinite line of telephone poles along the highway, they march off away from you, going on forever. And between each natural number and the next is a gap. In fact, that gap is what makes it meaningful to say "the next natural number."

Things are more challening for the rational numbers. First, can we even order them on a line? Yes. Like so.

Definition. We say that the rational number a/b is *larger* than the rational number c/d when

$$a \cdot d > c \cdot b.$$

Exercise 4. Find an example of rational numbers a/b and c/d where a/b is larger than c/d.

Exercise 5. Find an example of rational numbers a/b and c/d where a/b is smaller than c/d.

Exercise 6. Place all of the rational numbers you have found so far on a number line showing their relative sizes.

Exercise 7. Explain why the following is true:

> For each pair of rational numbers a/b and c/d, either a/b is larger than c/d, or a/b is equivalent to c/d, or a/b is smaller than c/d.

Density of the Rational Numbers

So we see that both the rational numbers and the natural numbers can be placed along a line. How are things really different for the rational numbers? Well, they keep popping up in different places.

Exercise 8. Find a rational number which is larger than 1/2 and smaller than 3/4.

Exercise 9. Find a rational number which is larger than 1/2 and smaller than 6/11.

Exercise 10. Find a rational number which is larger than 5/51 and smaller than 93/175.

Exercise 11. Suppose you are given a pair of rational numbers a/b and c/d. That is, you know they are rational numbers, but you do not know exactly what they are. We assume that a/b is smaller than c/d.

Show how to find a rational number x/y so that x/y is larger than a/b and smaller than c/d.

Challenges

Challenge. Use the definition of *infinite set* to prove that the set $\mathbb{Q}$ of rational numbers is an infinite set.

Challenge. Repeat the last exercise, but use a *different subset* of $\mathbb{Q}$ to help you do the job.

Introduction

In this reading, we explore the existence of numbers which are not rational numbers. We also introduce the idea of a "proof by contradiction."

Goals

At the end of this assignment, a student should be able to:

- Explain why $\sqrt{2}$ is not a rational number.
- Explain what a proof by contradiction is.

A student might also be able to:

- Give an argument that $\sqrt{3}$ is not a rational number.

Reading and Questions for Cantor's Paradise Meeting 11

We have already encountered many different types of numbers in our study. We have seen *natural numbers*, *integers*, and recently *rational numbers*. There are a great many rational numbers, and they are very useful. These days, school children spend a lot of time learning about these kinds of numbers, what they might represent, and how to work with them.

An Ancient Discovery

Of course, at one point in time, detailed knowledge of rational numbers was not so common. To the ancient Greek culture mathematicians like Pythagoras and Euclid, dealing with these kinds of quantities was advanced mathematics. They thought of rational numbers as *ratios*, especially ratios of geometric lengths.

There is an old story, which may or may not be true, that the mystical cult of *Pythagoreans* learned about a new kind of number, which was not expressable as a ratio. Much of this group's worldview was built around the idea that ratios were of a singular importance, so this new knowledge was shocking and unsettling. So unsettling, that the gods drowned the man on a sea voyage. You can find a synopsis of the story on Wikipedia[1].

For some time, the negative feeling about these quantities remained. There is still some trace of the negativity in the modern names: numbers expressable as ratios are called *rational numbers*, numbers which are not expressable as ratios are called *irrational numbers*.

What exactly did the ancient mathematician find? This poor soul learned that the diagonal of a square with side length equal to 1 is not expressable as a ratio. These days, that length is called $\sqrt{2}$, and we instead say this:

Theorem 1. The number $\sqrt{2}$ is not a rational number.

A Preface on the Argument

We are going to look at the proof of that theorem. It is not necessary to be afraid of the word *proof* here. All that we mean by the word *proof* is that we are going to give a convincing argument about why the theorem is true.

But in this case, the convincing argument has a clever technique in it. The argument is an example of a *proof by contradiction*. What happens is this:

Step 1: We begin by assuming the theorem is **false**.
Step 2: We use that assumption to argue that some other thing happens.
Step 3: The trick is that the "other thing" has to be obviously wrong.
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This is the logic behind a proof by contradiction. It can be easy to get things turned around a bit when you are new to this type of argument, because you have to assume negative things a lot. People who are used to this kind of argument spend all their time on Step 2, and they do not usually explain that the other steps are happening, too. But after a bit of practice, you will find this natural and all will be well.

Exercise 1. Have you ever seen a proof by contradiction before? Try to think of one or two situations where it has come up.

Exercise 2. How is a proof by contradiction like a jury trial? (This is a weak analogy, but there is something to be learned here.)

Three Facts we need for the proof

To make the proof run, we need three facts which you likely already know. So that things are clearest, we state them right now up front as things you can and should believe. But so we do not get distracted, we will not give proofs of these statements.

The Fundamental Theorem of Arithmetic

Recall that we have already seen this result. It will be important again below.

Theorem 2 (Fundamental Theorem of Arithmetic)**.** If n is a natural number, then there is a unique way to write n as a product of prime number factors.

There are actually two things happening here. First, every natural number can be written as a product of prime numbers. Second, there is only one way to write a given number as a product of prime numbers. The only thing you can do is rearrange the list of primes so it reads in a different order.

The Divisors of Squares Come in Pairs

Theorem 3 (Divisors of squares)**.** Let a be a natural number. If a number p is a prime factor of a^2, then p^2 is a factor of a^2.

Example 1. Consider the number $a = 45$. The prime factorization of $a = 45$ is $45 = 3 \cdot 3 \cdot 5$. So the prime factorization of $a^2 = (45)^2 = 2025$ is

$$(45)^2 = (3 \cdot 3 \cdot 5)^2 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5$$

Each prime factor 45 shows up **twice** in the list of prime factors of 45^2. This includes both 3's.

Rational Numbers in Lowest Terms

Theorem 4 (Lowest Terms)**.** Every rational number a/b can rewritten as an equivalent rational number c/d where c and d have no common prime number factors.

The special representation c/d is said to be in *lowest terms.*

Example 2. Consider 93/39. How may it be expressed in lowest terms? We factor the numerator and denominator and cancel common factors.

$$\frac{93}{39} = \frac{3 \cdot 31}{3 \cdot 13} = \frac{31}{13}$$

The equivalent rational number 31/13 is in lowest terms.

Exercise 3. Find a rational number which is equivalent to 465/225 and is in lowest terms.

Proof of the Theorem

By way of contradiction, assume that the theorem is false. That is, we assume that $\sqrt{2}$ is a rational number, and therefore expressable as one. This means that we can choose an integer a and a natural number b so that

$$\sqrt{2} = \frac{a}{b}. \tag{1}$$

By Theorem 4, we can make sure to clean things up and choose a and b so that the rational number a/b is in lowest terms. **This means that a and b have no prime number factors in common.**

Next, we will do a little algebra. We multiply Equation (1) through by b to clear the fraction, and then we square both sides of the result. Then we have this:

$$2b^2 = a^2.$$

Now think about the fact that the number on the right and the number on the left are equal. This means that the expressions of those numbers as products of prime number factors (from the Fundamental Theorem of Arithmetic) must be the same!

From what we see on the left-hand side, it is clear that 2 appears as a prime number factor in the list. Therefore, 2 must be a prime number factor of the right-hand side. That is, 2 is a prime factor of a^2. But if 2 is a prime divisor of a^2, Theorem 3 says $4 = 2 \cdot 2$ is a factor of a^2.

Now we go back in the other direction. Since $4 = 2 \cdot 2$ is a factor of a^2 it must be a factor of the left-hand side, too. But the left-hand side is $2b^2$. So we must have that 2 is a factor of b^2. But if 2 is a prime number factor of b^2, it must be a prime number factor of b.

Therefore, 2 is a prime number factor of a and 2 is a prime number factor of b.

This is an absurd situation. Our two bold statements contradict each other. Something must be wrong.

What is wrong? *Our initial assumption!* We conclude that $\sqrt{2}$ is not a rational number.

Some Challenges

Challenge. Adapt the argument above to prove that $\sqrt{3}$ is not a rational number.

Challenge. Adapt the argument above to prove that $\sqrt{6}$ is not a rational number.

Challenge. What happens when you try to adapt the argument to show that $\sqrt{4}$ is not a rational number? Where does the argument break?

References

[1] Wikipedia Entry on Hippasus, `http://en.wikipedia.org/wiki/Hippasus`, accessed 24 October, 2013.

Introduction

In this reading, we introduce a model for the *real numbers*. Also, we create a notational system for real numbers and explore some of its basic properties.

Goals

At the end of this assignment, a student should be able to:

- Give a definition of the set of real numbers.
- Given a real number, find the decimal notation for it.
- Given a decimal notation, find the associated real number.
- Describe the trouble with decimal notation.

Reading and Questions for Cantor's Paradise Meeting 12

So far we have encountered the natural numbers, $\mathbb{N}$, the integers, $\mathbb{Z}$, and the rational numbers, $\mathbb{Q}$. We have also encountered a thing which we want to be a number, $\sqrt{2}$, but is not a rational number. Somehow, we need a bigger set of numbers.

A hint lies in the way we usually arrange the numbers. We tend to imagine the natural numbers as being a long string of telephone poles along a country highway. Somehow, we have to imagine this set of things goes on "to infinity." The integers are similar, except they stretch out in both directions along the line.

The rational numbers are situated in the gaps. The number $1/2$ sits halfway between 0 and 1. The number $-4/3$ sits a third of the way from -1 to -2 (moving left this time).

Exercise 1. Draw a number line and place the numbers 0, 1, $1/2$, -1, -2, and $-4/3$ on it. Be sure to place them accurately.

But where shall we place a number like $\sqrt{2}$?

[Hang on to your hats, folks. This is what you learned in school, but it might not have been said quite this way.]

More Numbers

So, now we fix a special line, ℓ and two points O and I on that line.

Definition. A *real number* is a point on the line ℓ. The set of all real numbers is denoted $\mathbb{R}$.

That is a crazy definition, is it not? The geometric approach will make what we want to do much simpler. (On the other hand, it makes things like adding and multiplying much more challenging. That is not a concern for us, because we will not have much use for the arithmetic of real numbers.)

We are going to need a way to understand how to pick out particular real numbers and describe them.

Geometric Positional Notation: The Decimal System

Our system of notation for integers relies on using the ten symbols of the Hindu-Arabic numerals: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Now we shall develop a way use some notation to pick out a particular number. Since we have ten standard symbols, we shall use a geometric system based on dividing things into tenths.

To keep things simple we will work only with the set $\mathcal{R}$ of all real numbers which lie between O and I.

For each real number, that is for each point in the interval $\mathcal{R}$, we will assign a symbol of the form

$$0.a_1a_2a_3 \dots a_n \dots,$$

where each of the a_i's is one of allowed digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. The important thing here is that the symbol has as many a_i's as there are natural numbers!

Suppose one is given a point P lying in $\mathcal{R}$. We now describe the way to find the symbol for P.

First, divide $\mathcal{R}$ into ten subintervals of equal length. These subintervals are labelled with the ten digits of our numeral system, in order, from left to right.

Exercise 2. Draw a diagram of the interval R and its subdivision into ten equal length subintervals, with appropriate labels.

The point P lies inside one of these subintervals. The label on that subinterval is our a_1.

Now, we take that subinterval and further divide it into ten more equal length subintervals. Label these second level subintervals with the digits, in order, from left to right. We choose the number a_2 by whichever subinterval of this P lies in.

And this process continues over an over again, at each stage a new digit is chosen by which of the ten subintervals the point P lies in.

Example 1. If the point P lies in the third subinterval of $\mathcal{R}$ and then in the the fifth subinterval of that, then its decimal notation starts with 0.24.

Exercise 3. Draw a diagram that explains the last example.

Exercise 4. Suppose that the point P lies in the first subinterval of $\mathcal{R}$ and then in the tenth and last subinterval of that. Draw a diagram that explains why the decimal notation for P begins 0.09.

Exercise 5. Suppose that the point P lies in the third subinterval of $\mathbb{R}$, then the third subinterval of that, and then the third subinterval of that. What is the beginnig of the decimal notation for P?

Challenge. Dividing up $\mathcal{R}$ into tenths, and then dividing those intervals into tenths leads to a situation where $\mathcal{R}$ is divided up into one hundred equal length subintervals. If we label them by the numbers a_1a_2 from our system, what happens? Make a diagram and describe what you see.

Exercise 6. Find the complete decimal notation for the points O and I.

Finding a Given Decimal in $\mathcal{R}$

Of course, it is possible to go the other way, too. If you are given a real number already described in decimal notation, it is not too difficult to find where on the line it lives. Just work backward! The notation tells us how to successively locate which subintervals our point lies in. As we use more digits, we get more precision, because the subinterval shrink at each stage.

Example 2. A point with decimal notation $0.125\ldots$ lies in the sixth subinterval of the third subinterval of the second subinterval of $\mathcal{R}$.

Exercise 7. Make a diagram that explains the last example.

Exercise 8. Suppose that we divide $\mathcal{R}$ into one thousand subintervals of equal length. To which of these does a point with notation $0.125\ldots$ belong?

A Little Spot of Trouble

All of this works just great...expect for a little bit of trouble. What do we do about the points which lie in two intervals? For example, consider the midpoint M of $\mathcal{R}$, which lies halfway between O and I.

It is possible to see M as lying in *both* the fifth and sixth subintervals of $\mathcal{R}$, because it is the boundary between the two. Depending on the choice, we get different decimal notations for M. On one hand, we could start with $0.4\ldots$ and on the other we can start with $0.5\ldots$.

Suppose we choose the fifth subinterval. Then M is the farthest right hand point for the whole rest of the process. Every other choice we make is forced to be the tenth subinterval, so we get digits of 9. We see that M has decimal notation $0.49999999\ldots$ where the 9's repeat the rest of the way.

Instead, suppose we choose the sixth subinterval of $\mathcal{R}$. Then M is the farthest left hand point for the whole rest of the process. Every other choice we make is forced to be in the first subinterval, so we get digits of 0. We see that M has decimal notation $0.5000000\ldots$ where the 0's repeat the rest of the way.

Observation. The decimal notation of a point P in $\mathcal{R}$ can only be ambiguous if P lies on the boundary between two subintervals at some point in the process. In such a case, P will have **two** notations, one will end with an infinite string of 9's and the other will end with an infinte string of 0's.

Exercise 9. Consider the number Q which is the midpoint of the segment OM. This Q lies halfway between O and M, and has two decimal notations. Find these two notations.

Introduction

We discuss the relationship between rational numbers and real numbers.

Goals

At the end of this assignment, a student should be able to:

- Given a rational number, identify the real number to which it corresponds.
- Given a real number in decimal notation, decide if that number is a rational number or not.
- Given a real number which possibly comes from a rational number, find the relevant rational number.

A student might also be able to:

- Describe a few real numbers for which the decimal notations are difficult to determine exactly.

Reading and Questions for Cantor's Paradise Meeting 13

We have developed several number systems at this point, and at each stage things have gotten larger. If you look, the natural numbers have a natural inclusion into the integers. (Each number n basically gets sent to itself.) Similarly, the integers have a natural inclusion into the rationals. (An integer a gets sent to the rational number $a/1$.) In each of these inclusions, it is clear exactly which numbers come from the smaller set, and which ones do not.

But what about the rational numbers and the real numbers? We now describe the situation.

Identifying a Rational Number with a Real Number

Again, we shall focus our attention on numbers which lie between 0 and 1. Suppose we are given a rational number a/b in this interval. Then we know these things:

- a is not negative.
- $a \leq b$.

What real number should this correspond to? Well, geometrically, we divide the interval from O to I into b equal parts, and then we take the one a steps from the left. But that does give us the decimal notation all that easily.

How do we find the decimal notation? Suppose the decimal notation looks like

$$0.r_1r_2r_3r_4\ldots$$

Then r_1 marks the subinterval containing a/b in our system of even division into tenths. So we must have that r_1 is the digit that gives us

$$\frac{r_1}{10} \leq \frac{a}{b} < \frac{r_1+1}{10}.$$

This is equivalent to

$$r_1 \leq 10 \cdot \frac{a}{b} < r_1 + 1.$$

This means we want to choose the biggest digit r_1 so that $r_1 \leq 10 \cdot \dfrac{a}{b}$.

Once we have done so, we want to look at how much is left over:

$$x = 10\frac{a}{b} - r_1$$

This is a new rational number.

One crucial fact is this:

> **The number x is a rational number, which can be expressed having the same denominator b as a/b. So $x = a'/b$ for some other number a'.**

And x will be smaller than 1, because of the choices above. The key is that we now can find r_2 by starting over with the number x. And then you can find r_3 by starting over with whatever is left over from that.

Geometrically, we have replaced the "keep subdividing into tenths" business with the algebraic step of "scale everything out by a factor of ten." This has the same effect, but is more convenient for arithmetic.

Notice that the process has a few steps which end with "take the left-over part and repeat." This means that it can keep going, with no well-defined end in sight.

Example 1. Consider the rational number 5/6. We shall find the decimal notation for this as a real number.

First, we note that $10 \cdot \frac{5}{6} = \frac{50}{6}$ lies between 8 and 9, since $8 \cdot 6 = 48$. So $r_1 = 8$ and $x_1 = \frac{50}{6} - \frac{48}{6} = \frac{2}{6}$. We will start over with the new number 2/6. Note: We will not reduce this fraction. Things will be clearer for our "big picture" if we leave it alone.

Step Two: note that $10 \cdot \frac{2}{6} = \frac{20}{6}$ lies between 3 and 4, since $3 \cdot 6 = 18$. So $r_2 = 3$ and $x_2 = \frac{20}{6} - \frac{18}{6} = \frac{2}{6}$. Hey, wait! That is the same as before! This is now stuck and will keep repeating the value 3 for the digits r_i.

We conclude that $\frac{5}{6} = 0.8333333\ldots$ Writing a bunch of repeating threes is boring. Instead, let's put a bar over one three, and agree the bar means "repeat this stuff over and over forever."

$$\frac{5}{6} = 0.8\overline{3}.$$

We have now completely understood this one case.

Exercise 1. Try this basic process out with the number 7/8.

Exercise 2. Try this basic process out with the number 1/6.

Alright, now it is time for the big secret. All of that work can be organized into an algorithm that can be performed quickly and accurately as long as you do the book-keeping properly. That algorithm is called... **long division**.

Exercise 3. Find the decimal notation representations for 7/8 and 17/93. Compare your work to what you were doing above. Do you see how it is the same?

Which Real Numbers Are Rational Numbers in Disguise?

How can we tell if one of our real numbers is a rational number? The clearest way to figure this out is to observe closely what the process is, and remember not to change the denominators.

Let us begin with a rational number and see what is possible. If our input to the long division process is a/b, then at each step compute some value r_i and we replace the current problem with a new one, where the new input has the form a'/b. We repeat this over and over, and there are only b different options for values of the numerator, because the numerators come from this list:

$$0, 1, 2, 3, \ldots, b-2, b-1$$

So, somewhere along the line, but definitely by the time we have done the basic process b different times, we will get a repeat of a numerator we have seen before. At that point, we get that the whole process starts repeating.

Note: if you ever get $x = 0/b$, then you just get 0's from then on. 0's make life easier. In fact, you are probably used to just quitting after you get the first 0. It seems weird to write $1/2 = 0.5000000000\ldots$ *and keep writing all of the zeros.* Usually, people just stop and say that the decimal notation has *terminated.* By our original definition, the notation has all of those zeros in it! So 0 is a bit of a special case. But it is possible to have the numerators bounce around to all of the other $b-1$ choices before you get a repeat.

So, we just observe this:

Observation. If a real number is an expression of a rational number then its decimal notation is *eventually repeating.* That means that after some initial block of dancing around, we get some other block of digits which will repeat. Furthermore, the block of repeating digits of a number a/b is no longer than $b-1$ digits long.

Of course, the other direction is true, too. If the decimal notation for a number is eventually repeating, then that real number is a rational number.

The way to see this is to use a clever algebraic trick, which we now illustrate on the real number

$$y = 0.73\overline{25} = 0.732525252525252525\ldots$$

Example 2. First, multiply by 100 to move the decimal point two places. Then subtract out the part which does not repeat to focus our attention on the repeating bit. We shall just consider the number z given by

$$z = 100 \cdot y - 73 = 0.\overline{25} = 0.2525252525\ldots$$

We have done clean up to remove the non-repeating part for now. We will go back and take care of it in a minute. If we perform a similar trick with z, we can make this neat little bit of magic happen:

$$100 \cdot z - 25 = 0.\overline{25} = z$$

Hey, z fits in a linear equation! We can solve that to find $z = \frac{25}{99}$.

Finally, we put this information back in our expression for y

$$\frac{25}{99} = 100 \cdot y - 73$$

and solve for y

$$y = \frac{1}{100}\left(73 + \frac{25}{99}\right) = \frac{73 \cdot 99 + 25}{9900} = \frac{7252}{9900}.$$

That is clearly a rational number. If you are more comfortable, you can put this rational number into lowest terms

$$y = \frac{1813}{2475}.$$

Exercise 4. Using the above example as a guide, turn the following eventually repeating decimal notations into the standard form for rational numbers.

- $x = 0.47\overline{2}$. (*Hint: the answer is* 17/36.)
- $x = 0.\overline{428571}$

What about Real Numbers which are not Rational Numbers

It is convention to refer to a real number which is not a rational number as an *irrational number*. It is not a very nice name.

So, above we described which real numbers are rational numbers by using their decimal notations. What does this mean for the decimal notations of irrational numbers?

Challenge. What can you say about the decimal notation of the following numbers, which are known to be irrational? What property to they have in common?

$$\pi, \quad \sqrt{2}, \quad \sqrt{3}, \quad e$$

Introduction

We recap some of the different types of sets we have encountered, and try to group them by “size."

Goals

At the end of this assignment, a student should be able to:

- Name a variety of different types of sets.
- Explain clearly to someone else a surprising example of two sets that are really “the same size.”

Reading and Questions for Cantor’s Paradise Meeting 14

How many truly different sizes of sets have we encountered? There are a couple of important distinctions made in our work so far. Where do our various examples sit?

Finite Sets

Recall that a set is *finite* if its elements can be matched with the elements of some initial segment $\lfloor n \rfloor = \{1, 2, 3, \dots, n\}$ of the natural numbers.

The set $\{A, B, C\}$ has three elements and is finite. The set of all arrangements of the letters ABC is also finite, though this time there are six elements. The official roster for *Math 1100-03 Math in Decision Making, in Fall 2013* is a set M of students with 67 elements. The set

$$C = \{X \mid X \text{ is a two element subset of } M\}$$

is a bigger set, but it is still finite. In fact, C has $\frac{66 \cdot 67}{2} = 2211$ elements.

We have also come across the sets

$$B_5 = \{0/1 \text{ strings of length } 5\}$$

and

$$\mathcal{P}(\lfloor 5 \rfloor) = \{X \mid \; X \text{ is a subset of } \lfloor 5 \rfloor\}$$

and seen that these are the same size.

In fact, that power set construction is quite powerful, and can be used to make ever larger and larger sets. In general, if X has n elements, then $\mathcal{P}(X)$ has 2^n elements.

Infintite Sets

What examples of inifinite sets have we seen? A partial list looks like this:

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}^+, \mathbb{Q}, \mathcal{E}, \mathcal{O}, \mathcal{W}$$

For the naturals, the integers, the positive rationals, the rationals, the evens, the odds, and finally, the set of all "finite length mathematical words." These funny sets have the distinguishing characteristic that they are all the same size as $\mathbb{N}$. Such sets are called *countably infinite*. Of course, that means that in some sense all of these sets have the same size as each other.

Exercise 1. Find someone who isn't in this class, but could be, and explain to them how weird and wonderful it is that the sets $\mathbb{N}$ and $\mathbb{Q}$ have the same number of elements.

But we have encountered other infinte sets, too.

$\mathbb{R}$: the real numbers,
$\mathcal{R}$: the real numbers between 0 and 1,
B_∞ : the set of infinitely long 0/1 strings,
$\mathcal{P}(\mathbb{N})$: the collection of all subsets of the natural numbers,
$\mathcal{C}$: the set of all infinte words on the letters L and R, and
$\mathcal{I}$: the set of all infinite words on the letters l, c and r,
$\mathcal{I}_c$: the set of all infinite words on the letters l, c and r, but that do not use c.

Exercise 2. Are any of the sets in this last list provably of "the same size?" That is, can you find a matching between any pair of these? Go through what you have learned so far and sort out what is what.

Which sets have we not, yet, figured out size relative to some set we "understand?"